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Basic topics on damage pseudo-potentials

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This paper is dedicated to the memory of John B. Martin

Abstract

The meaning and the possible construction of the so-called damage potentials, used frequently in models describing the progressive failure of materials, are analyzed within the setting of a general framework for damage models. The treatment holds for both internal variable models (in which damage variables are considered as quantities that are not observable, and only kinetic equations describe their evolution) and a larger setting of multifield theories (in which damage variables satisfy appropriate balance equations of internal interactions). © 2001 Elsevier Science Ltd. All rights reserved.

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1. Introduction

In their books on the mechanics of damage phenomena, Krajcinovic (1996) and Lemaitre (1992) propose two different general forms of the so-called damage potentials:

In Krajcinovic's formulation, the damage potential is a real-valued functional of the affinities conjugated thermodynamically to the "cumulating damage" indicators. With respect to brittle processes arising in linearized deformation regime, such a potential may be expressed in his simplest quadratic form as

$$\Omega = \sum_{k=1}^{N} \Lambda_k D_k, \tag{1.1}$$

where Λ_k is the *affinity* associated to the kth *damage mode* D_k. For ductile materials, the expression of Ω needs to be modified in order to take into account the influence of plastic deformations on damage evolution.

Again in the case of linearized deformations, von Mises plastic behavior and damage evolution, Lemaitre proposes for the damage potential Ω the expression

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$$\Omega = \left(\boldsymbol{\tau}^D - \mathbf{X}^D\right)_{\mathrm{eq}} - R - \boldsymbol{\tau}_y + \frac{3}{4X_\infty}\mathbf{X}^D \cdot \mathbf{X}^D + \Omega_D(Y;(r,D)), \tag{1.2}$$

where $\boldsymbol{\tau}^D$ is the deviatoric stress tensor, $\mathbf{X}^D$, the kinematic hardening stress tensor, R, the isotropic hardening stress variable, Ω_D, the damage term of the dissipation potential, Y, the strain energy density release rate, r, the resistivity and, D, the damage variable.

Notwithstanding their generality, expressions (1.1) and (1.2) are only special cases of the whole range of possible candidates used by many authors to derive evolution laws of damage in terms of "normality rules of generalized standard materials" (see Krajcinovic (1996) and references therein) in analogy to the classical plasticity theory.

Of course, the determination of the evolution rule is a crucial point of the formulation of damage theories. In studying these rules, some questions arise:

- In which sense with the word "potential" may be used for a phenomenon, like damage evolution, which is dissipative?
- To which extent the usually postulated (generalized) normality rules may be justified in damage mechanics?
- Also, how damage (pseudo) potentials may in general be formulated independent of special assumptions of constitutive nature and how they can be related only to the characteristic features of all possible models, say the primitive concept of state, the action functional, the existence of a set of limit states with respect to some condition?

The above questions are of basic nature. More technical ones are related to the conditions of differentiability of possible general expressions of damage pseudo-potentials (when the existence is proven) and their lower semi-continuity which is useful to assure the validity of maximum and minimum principles, like the maximum dissipation principle. Answering all these questions is a delicate task owing to their basic nature and generality.

The present paper tries to indicate a possible way of discussion, following guidelines of a general framework for damage theories presented in Mariano and Augusti (1997) and Mariano (1997, 1998). Such a framework is discussed furthermore here, and the number of his basic axioms is reduced (as a first result), thus allowing to elude some questions that can arise about the existence and the physical sense of a natural topology in the state space of a damaging body.

Moreover, some concepts in Mariano and Augusti (1997) are rendered more specific here from the analytical point of view and not left to the physical intuition of the reader.

It is shown that a damage pseudo-potential may be defined as the infimum of the values of the action attained starting from each state in the admissibility region of the state space and going up to an ultimate failure set. Moreover, such a damage pseudo-potential is the largest one of all possible pseudo-potentials referred to the same set.

Convexity properties for the elastic admissible stress range, at a prescribed state of a given material patch, may be justified on the basis of a general inequality for the action functional which can be deduced from the basic axioms discussed.

2. Preliminary concepts and axioms

Damage is per se a relative concept. When a body in a given state is qualified as "damaged", one intends to say "damaged with respect to ...".

Usually, damage is identified with something (not always well defined) whose evolution induces loss of load-bearing capacity or serviceability of bodies. Thus, corrosion phenomena, or microcrack evolution, or interactions with them and other phenomena are related to damage evolution.

Consider for example a body identified in a given configuration with an open, simply connected set $\mathscr{B}_0$ of the three-dimensional Euclidean space. Consider also another configuration $\mathscr{B}_1$ obtained by applying to $\mathscr{B}_0$ some loading process that produces in $\mathscr{B}_0$ a certain set of microcracks. Commonly, $\mathscr{B}_1$ is considered damaged with respect to $\mathscr{B}_0$.

Moreover, it is possible to say that another configuration $\mathscr{B}_2$ is damaged with respect to $\mathscr{B}_1$ if

- there exists some ‘‘loading’’ process by which it is possible to reach $\mathscr{B}_2$ from $\mathscr{B}_1$ and the contrary is not possible;
- the set of microcracks of $\mathscr{B}_1$ is contained in the set of microcracks of $\mathscr{B}_2$.

Several alternatives exist for describing a damage state (i.e., for example, a microcrack system). The scientific literature is rich of choices of descriptors. Moreover, the kind of interactions between subbodies (associated to the descriptors) determines the representation of the power and of the evolution equations.

Owing to the varieties of proposals, when it is intended to indicate general properties that can be valid for a wide range of models, the characteristic objects which are common to each model need be usable (a priori) without the necessity of any special choice for their explicit representation. These objects are the *state*, the *action* functional (i.e. the power) and the loading or unloading *processes* that induces *state transformations*. The physical phenomena related to damage should be represented by these concepts.

More precisely, a damaged body, B, is usually described by associating to each material patch **P,** the position of its center of mass and information on the possible engines of damage evolution. Here, these engines are considered as systems of material defects within the body.

A *physical configuration* is thus a mapping k from the body B into the Cartesian product of the three-dimensional Euclidean space $\mathscr{E}^3$ and a certain manifold $\mathscr{M}$ (here, considered as finite dimensional and paracompact [1]). $\mathscr{M}$ is the collection of possible (or physically acceptable) defect configurations of each material patch. Therefore, k is defined by

$$k : \mathrm{B} \rightarrow \mathscr{E}^3 \times \mathscr{M}, \tag{2.1}$$

$$\mathbf{P}(\in \mathrm{B}) \overset{k}{\mapsto} (\mathbf{x}, \boldsymbol{\varphi}). \tag{2.2}$$

The mapping $\mathbf{x}(\cdot) = k_{\mathscr{E}^3}(\cdot)$ is the *apparent configuration*, or *placement*, of the body, while $\boldsymbol{\varphi}(\mathbf{P}) = \mathbf{k}_{\mathscr{M}}(\mathbf{P})$ is the order parameter, i.e. the descriptor of the configuration of the defects within the patch.

In general, $\mathscr{M}$ is a non-linear manifold. [2] Moreover, it cannot be Riemannian.

As a typical special choice, in the case of isotropic damage, $\mathscr{M}$ coincides with the interval [0, 1] of the real line. Other choices can be made. in particular, $\boldsymbol{\varphi}$ may be a vector field or a second-order tensor-valued field, or higher-order tensor-valued field, with different meanings. The geometrical properties of $\mathscr{M}$ are crucial in building up a certain model.

First of all, the metric defined on $\mathscr{M}$ is crucial in the representation of the kinetic energy (if any) that can be associated to $\boldsymbol{\varphi}$, depending on the physical significance of $\boldsymbol{\varphi}$ itself.

Moreover, by using Eq. (2.1), one intends to consider $\boldsymbol{\varphi}$ as an *observable* quantity to which interactions performing explicit working need be associated and thus balanced (see Section 4). But, these interactions

[1] Even if, in general, $\mathscr{M}$ may be infinite dimensional, say for example, a Banach manifold.

[2] Though by Withney's theorem $\mathscr{M}$ may be embedded in some linear space, the embedding is not unique and its physical meaning changes time to time. So, in general, an embedding in a linear space cannot be used a priori.

(namely substructural interactions determined by the various damage microstructural situations or modes) need be represented in some way. When it is possible to evaluate in a covariant manner (thus with absolute meaning) the gradient of $\boldsymbol{\varphi}$, these interactions may be represented through appropriate tensors usually called microstresses and self-forces, in analogy (or better in assonance) to the usual terminology in mechanics. So, the order parameter takes the meaning of a self-compatible (and thus self-interacting) field. Consequently, the interactions associated to $\boldsymbol{\varphi}$ satisfy appropriate balance equations; the model becomes a *multifield damage model* (see Augusti and Mariano (1999) and references therein) which allows to elude some problems of ill-posedness and mesh dependence, related traditionally to classical damage models.

When a physically reasonable connection cannot be evaluated on $\mathscr{M}$, these interactions are represented by certain general functionals (Capriz and Giovine, 1999).

Alternatively, one can decide to adopt the classical representation $k : \mathrm{B} \rightarrow \mathscr{E}^3$, change Eq. (2.1) and consider the descriptor $\boldsymbol{\varphi}$ of damage modes as an internal variable (which is non-observable by definition). In this case, only thermodynamic affinities (not genuine interactions) can be associated (Coleman and Gurtin, 1967). Such affinities do not perform explicit power. This is true almost in all classical damage theories (Lubarda and Krajcinovic, 1995; Simo and Ju, 1987).

Of course, the subdivision presented above is very rough. In fact, not only the above mentioned subclasses may overlap, [3] but other tracks may be in turn followed.

As a typical example, variational minimum problems of non-convex energies of standard Cauchy nonlinear elastic bodies lead to solutions whose strong irregularities may be interpreted as symptoms of the occurrence of local brittle failure modes (Choski et al., 1998).

As it has been roughly mentioned above, different basic starting points can be adopted to formulate continuum models of damaged bodies. An analogous variety of starting points may be recognized even in models based on percolation of breaking bonds in lattice systems, which are typical of molecular dynamics simulations (Hermann and Roux, 1990; Krajcinovic et al., 1998).

In any case, one should recognize as a primitive concept the *state*, σ, of a body as the collection of *fields* describing it, namely deformation gradient $\mathbf{F}$, order parameter $\boldsymbol{\varphi}$, temperature ϑ, etc. So that

$$\sigma \equiv (\mathbf{F}(\,\cdot\,), \vartheta(\,\cdot\,), \boldsymbol{\varphi}(\,\cdot\,), \nabla\boldsymbol{\varphi}(\,\cdot\,), \ldots). \tag{2.3}$$

The state σ is an element of the *state space* Σ. Since σ has been defined as a collection of fields (so it is thought as describing the *whole* body), Σ is *infinite dimensional*. In the case in which the state is simply referred to a material patch (*substantial point* in Truesdell's words), σ is not a collection of fields; so Σ is finite dimensional. When memory effects, e.g., are considered at each material patch, Σ is once more infinite dimensional.

Except in cases in which it will be explicitly declared, Σ will be referred to as the whole body throughout the paper.

Incidentally, in such a setting, the stress mapping 'Str' is defined on Σ and associates to each state the collection of classical or generalized (Capriz, 1985) measures of internal interactions that appear in the special model considered time to time.

A basic theoretical problem in characterizing Σ is the possibility of recognizing, among the possible ones, the natural topology on it. Such a problem could look apparently trivial but it is not so, because a topology makes sense to operations and/or properties with crucial physical meaning like for example the continuity of the power, i.e. the action functional, and so on. To reach, e.g., the minimum of the energy or the maximum of the dissipation, it is necessary to recognize lower semicontinuity properties for the relevant functionals. However, such properties can exist in a given topology and cannot exist in another one.

[3] Really, standard internal variable models can be obtained from a choice of the type of Eq. (2.1) by appropriate *internal constraints*. In this case, $\boldsymbol{\varphi}$ becomes the descriptor of a *latent* microstructure (in the sense used by Capriz (1985)).

For simple materials, Noll (1972) suggested that, in the case in which Σ is finite dimensional (then referred to a single material patch), the natural topology on it is given by the coarsest uniformity that makes uniformly continuous the stress mapping Str. For more complicated descriptions of bodies, and especially for infinite dimensional state spaces, the stress mapping may take values into spaces that are, in general, different from the linear space of symmetric bilinear forms as in the case of simple materials. Such spaces may be sets of distributions, stochastic processes and so on. In this case, it is not obvious what the natural uniformity (which is the precursor of the topology) is in both the domain and the range of Str.

Note that the results on the foundations of damage mechanics in Mariano and Augusti (1997) suppose the existence of an underlying topology in the state space. Consequently, from the considerations above we realize the need of rendering these results independent of the choice of the topology: this is a goal of this paper.

Let 'Dur' be the set of bounded intervals of $\mathscr{R}^+$ of the form

$$[0, d_{\mathrm{P}}]$$

with d_{P} as some positive real number, $0 < d_{\mathrm{P}} < +\infty$.

Moreover, let $\mathscr{T}$ be the set of all paths in Σ generated considering the elements of Dur, namely

$$\mathscr{T} \equiv \{\rho | \rho : I \to \Sigma, I \in \mathrm{Dur}\}. \tag{2.4}$$

A *process* P is a mapping defined as

$$\mathrm{P} : D \times \mathrm{Dur} \to \mathscr{T}, \tag{2.5}$$

$$D \subseteq \Sigma,$$

where $D \equiv D(\mathrm{P})$ is the domain in $\mathscr{A}$ of the process P.

In the following, $\mathscr{T}^*$ represents the set of paths associated to processes, i.e.

$$\mathscr{T}^* = \{\rho | \exists \mathrm{P} \in \Pi \quad \text{s.t. } \rho(0) \in D(\mathrm{P}),\ \rho(t) \equiv \rho_{\mathrm{P}^t},\ \forall t \in I,\ I \in \mathrm{Dur}\}. \tag{2.6}$$

Of each path generated by P and indicated by $\rho_{\mathrm{P}^t}\sigma$, σ is the *initial state*, $\rho_{\mathrm{P}}\sigma$, the *final state* attained at $t = d_{\mathrm{P}}$, where d_{P} is the *duration* of the process, while $\rho_{\mathrm{P}^{\bar{t}}}\sigma$ (or with 't' otherwise marked) represents a given special state along the path. The path $\rho_{\mathrm{P}^t}\sigma$ is also called *state transformation*. Π is the set of *all* possible processes P. It is possible to define the *composition* of processes through a mapping

$$\Pi \times \Pi \to \Pi \tag{2.7}$$

which associates to each pair (P', P'') such that

$$\rho_{\mathrm{P}'}(D(\mathrm{P}')) \cap D(\mathrm{P}'') \neq \varnothing. \tag{2.8}$$

The resulting process $\mathrm{P}''\mathrm{P}'$ is such that

$$D(\mathrm{P}''\mathrm{P}') \equiv \rho_{\mathrm{P}'}^{-1}(\rho_{\mathrm{P}'}(D(\mathrm{P}')) \cap D(\mathrm{P}'')), \tag{2.9}$$

$$\rho_{\mathrm{P}''\mathrm{P}'}\sigma = \rho_{\mathrm{P}''}\rho_{\mathrm{P}'}\sigma. \tag{2.10}$$

In other words, $\mathrm{P}''\mathrm{P}'$ *is the process resulting from the successive application of* P' *and* P'' and $\rho_{\mathrm{P}''^t}(\rho_{\mathrm{P}'}\sigma)$ *is the continuation of the path* $\rho_{\mathrm{P}'^t}\sigma$ under the process P'' (Coleman and Owen, 1977). The process P might be interpreted as a load process, temperature and/or strain induced histories, order parameter induced histories and so on. In other words, P represents the interaction of the external environment with the body, inducing a state transformation.

As a simple example, consider a purely elastic beam clamped at one of its ends and load it with a time-dependent force applied at the other end in a direction which is orthogonal to the beam axis. The process

P is the application of the force during a time interval $[0, d_{\mathrm{P}}]$ and the state transformation is the successive deformation of the beam during the same time interval.

In the present paper, an *unload process* is defined as a process P^{u} such that

$$\mathrm{P}^{\mathrm{u}} : D \times \mathrm{Dur} \to \mathscr{T}, \tag{2.11}$$

$$\mathrm{P}^{\mathrm{u}}\mathrm{P}^{\mathrm{u}} = \mathrm{P}^{\mathrm{u}}. \tag{2.12}$$

No state transformation inducing ‘‘relaxation’’ phenomena (Noll, 1972, p. 22) is considered in this paper.

Given two arbitrary states σ_1 and σ_2 belonging to Σ, it is not always possible to recognize a process P such that $\rho_{\mathrm{P}}\sigma_1 = \sigma_2$. Of course, gluing or other forms of processes representing physical restoration procedures are not considered here. It is only assumed that there exist only some states, called *base states*, σ_{b}, from which it is possible to attain all states in Σ:

$$\forall \sigma \in \Sigma, \quad \exists \mathrm{P} \in \Pi \quad \text{s.t.} \quad \rho_{\mathrm{P}}\sigma_{\mathrm{b}} = \sigma. \tag{2.13}$$

Of course, a process P* such that $\rho_{\mathrm{P}^*}\sigma = \sigma_{\mathrm{b}}$ may not exist unless σ is a base state.

At this point, it is possible to introduce some axioms relevant to damage theories which have to be considered as additional to the classical ones of continuum mechanics (see Noll (1972) and references therein). These axioms are *revisited* versions of the ones presented in Mariano and Augusti (1997) with respect to which their number is reduced. Whereas in the quoted paper, the axioms used to establish a common framework for damage theories consider some sort of topology (the set of admissible states is there considered as a closed set in Σ), here, such a reference to an underlying topology is absent; thus, weak forms of approachability between states are not considered. [4]

A1 (*admissibility set*): There exists two sets $\mathscr{A}$ and $\mathscr{G}$ in Σ such that

- $\mathscr{A} \neq \emptyset$; $\mathscr{G} \neq \emptyset$;
- $\mathscr{A} \cap \mathscr{G} \equiv \emptyset$;
- if $\exists t_1$ such that $\rho_{\mathrm{P}^{t_1}}\sigma \in \mathscr{A}$, then $\rho_{\mathrm{P}^{t<t_1}}\sigma \subset \mathscr{A}$;
- if $\exists t_2$ such that $\rho_{\mathrm{P}^{t_2}}\sigma \in \mathscr{G}$, then $\rho_{\mathrm{P}^{t>t_1}}\sigma \subset \Sigma \setminus \mathscr{A}$;
- if $\exists t^*$ such that $\rho_{\mathrm{P}^{t^*}}\sigma \in \Sigma \setminus \mathscr{A}$, then $\exists \mathrm{t}'$ s.t. $\rho_{\mathrm{P}^{\mathrm{t}<\mathrm{t}'}}\sigma \in \mathscr{A}$ and $\rho_{\mathrm{P}^{\mathrm{t}>\mathrm{t}'}}\sigma \subset \Sigma \setminus \mathscr{A}$;
- $\forall \sigma \in \mathscr{A}$, $\exists \mathrm{P} \in \Pi$ such that $\rho_{\mathrm{P}}\sigma \in \mathscr{G}$.

A2 (*base states*): $\mathscr{A}$ contains (among others) all base states in Σ.

Of course, if σ_{b} and σ'_{b} are two base states, by definition, at least two processes P and P′ exist such that $\rho_{\mathrm{P}}\sigma_{\mathrm{b}} = \sigma'_{\mathrm{b}}$ and $\rho_{\mathrm{P}'}\sigma'_{\mathrm{b}} = \sigma_{\mathrm{b}}$. Moreover, it is not a great restriction to assume here that all base states may be obtained one from another by reversible state transformations.

In the present setting, a process P induces a *reversible* state transformation, starting from a given state $\sigma \in D(\mathrm{P})$, if there exists another process P* such that

$$\rho_{\mathrm{P}}(D(\mathrm{P})) \equiv D(\mathrm{P}^*), \tag{2.14}$$

$$\forall \bar{t} \in [0, d_{\mathrm{P}}], \qquad \rho_{\mathrm{P}^{\bar{t}}}\sigma \equiv \rho_{\mathrm{P}^{*(d_{\mathrm{P}}-\bar{t})}}\rho_{\mathrm{P}}\sigma. \tag{2.15}$$

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[4] If a physically significant topology in Σ exists, a state σ' may be considered approachable from another state σ if there exists a process P such that $\rho_{\mathrm{P}}\sigma \in \mathscr{O}(\sigma')$, where $\mathscr{O}(\sigma')$ is a neighborhood of σ' in some topology (Coleman and Owen, 1974).

A1 is a relaxed version of the axiom of *closure of the admissible set of states* and of the axiom of *possibility of damage* (Mariano and Augusti, 1997).

When a physically significant topology may be recognized in Σ, the set $\mathscr{A}$ may be considered as an open set in Σ. Thus, $\mathscr{G}$ is the set of the adherence points of $\mathscr{A}$. If $\mathscr{A}$ is a regular open set in the state space, i.e. it coincides with the interior of its closure, $\mathscr{G}$ is the boundary of $\mathscr{A}$. Such a special situation is typical of damage theories, where $\mathscr{G}$ is the failure surface of some admissibility set, but it is also typical of plasticity theory.

The picture presented by previous axioms is more general than the one of usual damage or plasticity theories when admissibility sets are selected in them. Typically, when in these theories $\mathscr{G}$ represents the boundary of some regularly open admissibility set, only reversible state-transformations are possible inside it and the first mechanism of failure and/or of plasticization occurs at $\mathscr{G}$.

Within the setting presented here, irreversible state transformations are possible in $\mathscr{A}$ or reversible ones.

To have an example, consider in the plane xy, a two-dimensional brittle elastic body made by the strip $(-\infty, +\infty) \times (-w, w)$, where $(-w, w)$ belongs to the y-axis and $w > 0$. For such a body, $\mathscr{G}$ can be chosen in different ways. Some of them can be (roughly speaking) the following:

- $\mathscr{G}$ may be the collection of states associated to the first apparition of microcracks;
- elements of $\mathscr{G}$ may be all states representing the existence of cracks whose projection on the y-axis is a segment contained within the interval $(-w^*, w^*) \subset (-w, w)$;
- finally, $\mathscr{G}$ may represent the set of states in which cracks cut completely the strip.

In the first case, only reversible (e.g. elastic) state transformations are possible in $\mathscr{A}$, thus the treatment reduces formally to usual cases of damage mechanics and plasticity. In the last two cases, irreversible processes are possible in $\mathscr{A}$. In this sense, $\mathscr{G}$ is only a *remote horizon* with respect to which some comparisons need to be made.

Up to this point, only states and processes have been considered, but they are not the sole basic tools of mechanical models. Another basic tool is the *action functional*, i.e. a real-valued functional which associates to each path generated by a process in the state space the power performed by the mechanical system on it. Of course, the explicit representation of the action depends on the special model selected time to time.

Indicating with $(\Pi \Diamond \Sigma)_{\text{fit}}$ the set

$$(\Pi \Diamond \Sigma)_{\text{fit}} = \{(\mathrm{P}, \sigma) | \sigma \in D(\mathrm{P}), \sigma \in \Sigma, \mathrm{P} \in \Pi\}. \tag{2.16}$$

Coleman and Owen (1974) defined the *action* $a(\cdot,\cdot)$ as a real-valued functional

$$a : (\Pi \Diamond \Sigma)_{\text{fit}} \to \mathscr{R} \tag{2.17}$$

such that

- the action is *additive* on processes, i.e.

$$a(\mathrm{P}''\mathrm{P}', \sigma) = a(\mathrm{P}', \sigma) + a(\mathrm{P}'', \rho_{\mathrm{P}'}\sigma). \tag{2.18}$$

- $a(\mathrm{P},\cdot)$ is continuous on states.

Really, the action may also be defined on the space of paths associated to processes, namely

$$a : \mathscr{T}^* \to \mathscr{R}. \tag{2.19}$$

In the present paper, it is only required the additivity of the action on processes, i.e. the additivity on paths owing to Eq. (2.10), and not *the continuity on states* because the concept of continuity implies a topology. The requirement of additivity on paths is simply due to the intrinsic nature of the action which is substantially a generalized form of *path-integral*.

On the actions, another axiom need be considered, namely

A3 (*inf-boundedness*): There exists at least one action $a(\cdot,\cdot)$ such that

$$|\inf_{\mathrm{P}\in\Pi}\{a_{\sigma\in\mathscr{A}}\to\mathscr{G}\}|<\infty. \tag{2.20}$$

In A3, the symbol $\{a_{\sigma\in\mathscr{A}}\to\mathscr{G}\}$ indicates *the set of all values reached by the action on all paths starting from* σ *and ending in* $\mathscr{G}$, namely

$$\{a_{\sigma\in\mathscr{A}}\to\mathscr{G}\}\equiv\{a(\mathrm{P},\sigma)|\sigma\in D(\mathrm{P}),\rho_{\mathrm{P}}\sigma\in\mathscr{G}\}. \tag{2.21}$$

A3 states that the set $\{a_{\sigma\in\mathscr{A}}\to\mathscr{G}\}$ is *bounded from below*. In other words, it is required that the minimum amount of energy necessary to bring a given admissible state to the remote horizon of ultimate failure is finite.

On the basis of previous concepts it is possible to affirm that a process P induces damage (with respect to a given action satisfying A3) starting from a given state σ belonging to $\mathscr{A}$ iff

- $\exists\mathrm{P}^*$ such that $\rho_{\mathrm{P}^*}\rho_{\mathrm{P^uP}}\sigma\in\mathscr{G}$,
- $\inf_{\mathrm{P}\in\Pi}\{a_\sigma\to\mathscr{G}\}>\inf_{\mathrm{P}\in\Pi}\{a_{\rho_{\mathrm{P^uP}}\sigma}\to\mathscr{G}\}$,

where $\mathrm{P^uP}$ represents the composition of the process P with an unloading process.

In other words, roughly speaking, the body is first loaded, then it is unloaded; so, comparison between the initial state and the final state, after unloading, is made in terms of energy. Previous definition of *damaging processes* does not take into account different modes of damage but considers damage only in terms of power spent along state transformations.

In the following, Π° will represent the set of all damaging processes with respect to a given action. With reference to processes inducing damage, another axiom need to be introduced, namely

A4: If there exists a path in $\mathscr{A}$ induced by a damaging process that connects two different states, then these states can be connected only by paths induced by damaging processes.

Within the present setting, a *weak lower potential* (or *pseudo-potential*) for a given action a, in the admissibility set $\mathscr{A}$, is a real-valued function Γ defined as follows:

- $\Gamma:\mathscr{A}\cup\mathscr{G}\to\mathscr{R}$;
- $\forall\sigma_1,\sigma_2\in\mathscr{A}\cup\mathscr{G}$, if $\exists\mathrm{P}\in\Pi$ such that $\rho_{\mathrm{P}}\sigma_1=\sigma_2$, then

$$\Gamma(\sigma_1)-\Gamma(\sigma_2)\leqslant a(\mathrm{P},\sigma_1). \tag{2.22}$$

Moreover, a pseudo-potential Γ is said to be *referred to* $\mathscr{G}$ if it takes a constant value (indicated here with $\Gamma_{\mathscr{G}}$) greater than or equal to zero on $\mathscr{G}$. Thus, trivially

$$\forall\sigma\in\mathscr{A},\qquad\Gamma(\sigma)=\Gamma^*(\sigma)-\Gamma_{\mathscr{G}} \tag{2.23}$$

with $\Gamma_{\mathscr{G}}\geqslant 0$.

The axioms discussed in this section are tools to build up consistent models of damage evolution, in particular, models of continuum damage mechanics.

3. Some propositions from previous axioms

Proposition 1. *Given an action a satisfying A3,*

- $a(\cdot,\cdot)$ *is bounded from below when it is calculated on paths in* $\mathscr{A}$;
- $\forall \mathrm{P} \in \Pi^{\circ}$, $\forall \sigma \in \mathscr{A}$,

$$a(\mathrm{P}^{\circ}\mathrm{P}, \sigma) > 0. \tag{3.1}$$

Proof. Let σ be some element of $\mathscr{A}$, if P is such that $\rho_{\mathrm{P}}\sigma \in \mathscr{A}$, then by A1 there exist at least P′ and P″ such that $\rho_{\mathrm{P}'}\sigma \in \mathscr{G}$ and $\rho_{\mathrm{P}''}\rho_{\mathrm{P}^{\circ}\mathrm{P}}\sigma \in \mathscr{G}$ and the whole path $\rho_{\mathrm{P}^{t}}\sigma$ belongs to $\mathscr{A}$. Moreover, by triangular inequality

$$\inf_{\mathrm{P}}\{a_{\sigma} \to \mathscr{G}\} \leqslant a(\mathrm{P},\sigma) + \inf_{\mathrm{P}}\{a_{\rho_{\mathrm{P}}\sigma} \to \mathscr{G}\}. \tag{3.2}$$

Thus, simply

$$a(\mathrm{P},\sigma) \geqslant \inf_{\mathrm{P}}\{a_{\sigma} \to \mathscr{G}\} - \inf_{\mathrm{P}}\{a_{\rho_{\mathrm{P}}\sigma} \to \mathscr{G}\} \tag{3.3}$$

By A3, the difference

$$\inf_{\mathrm{P}}\{a_{\sigma} \to \mathscr{G}\} - \inf_{\mathrm{P}}\{a_{\rho_{\mathrm{P}}\sigma} \to \mathscr{G}\}. \tag{3.4}$$

is bounded, thus the action itself is bounded from below. In Eqs. (3.2)–(3.4) and in the following developments, $\inf_{\mathrm{P}}$ means $\inf_{\mathrm{P}\in\Pi}$. On the other hand, if P induces damage, the difference (3.4) is greater than zero, thus the second item of Proposition 1 holds true. □

The property of lower boundedness for the action implies the lower boundedness of the energy which is crucial in deducing balance equations (Šilhavý, 1989, 1997).

The inequality (3.1) justifies the use of work inequalities in damage theories, in analogy to plasticity, in order to obtain convexity of the stress range associated, for a given state σ, to the set of states that may be reached by σ using reversible state transformations. Obviously, such convexity properties are crucial in practical numerical applications when the state space Σ is referred to a single material patch of the body, whereas in this case, Σ is finite dimensional and Str takes values into a linear space.

Inequality (3.1) is general and is independent of the specific choice of a given model. Consequently, it holds true in both the case of usual internal variable models and the case of multifield descriptions of damaged bodies (for a simple application see Mariano (1999)).

Proposition 2. *The state function*

$$\chi : \mathscr{A} \cup \mathscr{G} \to \mathscr{R}^{+}, \tag{3.5}$$

$$\chi(\sigma) = \inf_{\mathrm{P}}\{a_{\sigma\in\mathscr{A}\cup\mathscr{G}} \to \mathscr{G}\}. \tag{3.6}$$

is a weak lower potential (*or pseudo-potential*) *in* $\mathscr{A}$.

Proof. By A1, $\forall \sigma \in \mathscr{A}$ there exists at least one P* such that $\rho_{\mathrm{P}^*}\sigma \in \mathscr{G}$. As a consequence of triangular inequality (3.2) and A3,

$$a(\mathrm{P}^*, \sigma) > 0. \tag{3.7}$$

Thus, χ takes values in $\mathscr{R}^{+}$. Moreover, let P be an arbitrary process inducing a state transformation in $\mathscr{A}$, i.e.

$$\rho_{\mathrm{P}}\sigma \in \mathscr{A}, \quad \forall \sigma \in \mathscr{A} \cap D(\mathrm{P}). \tag{3.8}$$

By triangular inequality (3.3), A1 and Eq. (3.6), the following inequality holds true:

$$\chi(\sigma) - \chi(\rho_{\mathrm{P}}\sigma) \leqslant a(\mathrm{P},\sigma). \qquad \square \tag{3.9}$$

Proposition 3. *The state function χ is the largest one of all possible pseudo-potentials referred to $\mathscr{G}$.*

Proof. In analogous way to the previous proposition, the proof starts by considering that by A1, $\forall \sigma \in \mathscr{A}$, there exists at least one P^* such that $\rho_{\mathrm{P}^*}\sigma \in \mathscr{G}$. Thus, given a pseudo-potential Γ,

$$\Gamma(\sigma) - \Gamma_{\mathscr{G}} \leqslant a(\mathrm{P}^*, \sigma) \tag{3.10}$$

Without loss of generality, $\Gamma_{\mathscr{G}}$ may be taken equal to zero. Consequently,

$$\Gamma(\sigma) \leqslant a(\mathrm{P}^*, \sigma) \tag{3.11}$$

and

$$\Gamma(\sigma) \leqslant \chi(\sigma). \qquad \square \tag{3.12}$$

.

Previous propositions give a first attempt to characterize the nature of damage pseudo-potentials and assure that the set of such potentials is not empty.

Note that the admissibility set $\mathscr{A}$ may be decomposed in a way such that

$$\mathscr{A} = \bigcup_c \mathscr{A}^c_\chi, \tag{3.13}$$

where

$$\mathscr{A}^c_\chi = \{\sigma \in \mathscr{A} | \chi(\sigma) = c, \ c = \mathrm{cost}\} \tag{3.14}$$

are pseudo-potential damage level sets.

4. Some final remarks

Previous general statements need to be specified when special models of damage mechanics are under examination.

Consider for example a solid whose behavior is linear elastic and the damage evolution is recognized through the degradation of the elastic constants (Simo and Ju, 1987). Let α be some descriptor of the damage state. Here, for simplicity, α is considered as a scalar varying within the interval [0, 1]; thus, only the isotropic damage can be evaluated. In this case, and with reference to a material patch, without considering temperature effects, the state σ is the pair $(\mathbf{E}, \alpha)$, where $\mathbf{E}$ is the strain tensor. Now, the stress mapping associates to $(\mathbf{E}, \alpha)$ the Cauchy's stress in a way such that

$$\mathrm{Str}(\sigma) = \mathbf{S}(\mathbf{E}, \alpha) = \mathbf{C}(\alpha)\mathbf{E}, \tag{4.1}$$

where $\mathbf{C}(\alpha)$ is the elastic tensor, depending on α.

Moreover, the evolution of α is ruled by

$$\dot{\alpha} = \langle 1 \rangle f(\alpha), \tag{4.2}$$

where $\langle 1 \rangle$ is equal to zero whereas some failure criterion is not violated and is equal to 1 in the opposite case, and f is obtained from damage potentials.

Now, the action functional at a material patch is here

$$\int_0^{d_P} \mathbf{S} \cdot \dot{\mathbf{E}} \, dt, \tag{4.3}$$

where d_P is the duration of the loading process $(\mathbf{b}(t), \mathbf{f}(t)) \equiv P$, with **b** representing volume forces and, **f**, applied surface tractions. [5] Of course, other choices of α can be made (Krajcinovic, 1996).

In the case of multifield description of damaged solids, let $\boldsymbol{\varphi}$ be the order parameter (i.e. an element of the manifold $\mathscr{M}$ quoted in Section 2): the state σ is now the three-plet $(\mathbf{F}, \boldsymbol{\varphi}, \nabla\boldsymbol{\varphi})$, where **F** is the gradient of deformation. As typical in multifield models, the stress mapping is such that

$$\mathrm{Str}(\sigma) = (\mathbf{T}, \mathbf{S}, \mathbf{z}). \tag{4.4}$$

In Eq. (4.4), **T** is the Piola–Kirchhoff stress tensor, **S,** the microstress tensor, and **z** measures self-forces (Capriz, 1989).

The action functional at a given material patch is now given by

$$\int_0^{d_P} (\mathbf{T} \cdot \dot{\mathbf{F}} + \mathbf{z} \cdot \dot{\boldsymbol{\varphi}} + \mathbf{S} \cdot \nabla\dot{\boldsymbol{\varphi}}) \, dt, \tag{4.5}$$

where d_P is the duration of the "loading process" $(\mathbf{b}(t), \mathbf{f}(t), \boldsymbol{\tau}(t))$. $\boldsymbol{\tau}$ represents boundary data on the interactions associated to the order parameter. [6]

Here, the analogous expression of Eq. (4.2) is given by

$$\langle 1 \rangle h(\sigma; \dot{\boldsymbol{\varphi}})\dot{\boldsymbol{\varphi}} = \mathrm{Div}\,\mathbf{S}(\sigma) - \mathbf{z}(\sigma), \tag{4.6}$$

where h is given by constitutive prescription. Different from Eq. (4.2), Eq. (4.6) has the meaning of a balance of interactions.

Convexity properties of the stress range allow the development of reliable numerical procedures and to obtain flow rules as normality conditions, analogous to the plasticity theory.

When a certain type of model is formulated, in both the setting of internal variables and of multifield approaches, to obtain convexity properties of the stress range, it is necessary to *assume* that Eq. (4.2) or (4.6) are greater than or equal to zero, once some choice of σ or $\boldsymbol{\varphi}$ has been made. Obviously, the model obtained depends on the validity of such a kind of assumption.

Proposition 1 clarifies this question because it assures the validity of the quoted inequalities, for every choice of σ or $\boldsymbol{\varphi}$, provided axioms 1–4 are satisfied.

Moreover, Proposition 2 assures the possibility of evaluating damage pseudo-potentials once an explicit expression of the action functional is selected. $\chi(\sigma)$ can be evaluated explicitly by using, e.g., some techniques that can be found in Owen (1984) where they are applied to elastic–plastic materials.

Finally, $\chi(\sigma)$ is also an upper bound for all possible pseudo-potentials that can be introduced for convenience in some of the quoted approaches to damage (Proposition 3).

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[5] Here, for simplicity, only a traction problem is considered. Assigned displacements at the boundary of the body are not taken into account.

[6] More precisely, $\boldsymbol{\tau} = \mathbf{sn}$, where **n** is the normal at the boundary of the body.

References

Augusti, G., Mariano, P.M., 1999. Stochastic evolution of microcracks in continua. Comp. Meth. Appl. Mech. Engng. 168, 155–171.

Capriz, G., 1985. Continua with latent microstructure. Arch. Rational Mech. Anal. 90, 43–56.

Capriz, G., 1989. Continua with Microstructure. Springer, Berlin.

Capriz, G., Giovine, P., 1999. Weakly nonlocal effects in mechanics. Quaderni del Dipartimento di Matematica dell'Università di Pisa, Sezione di Fisica-Matematica, Pisa.

Choksi, R., Del Piero, G., Fonseca, I., Owen, D.R., 1998. Structured deformations as energy minimizers in models of fracture and hysteresis. CMU Scientific Reports, 98-NA-002.

Coleman, B.D., Gurtin, M.E., 1967. Thermodynamics with internal state variables. J. Chem. Phys. 47, 597–613.

Coleman, B.D., Owen, D.R., 1974. A mathematical foundation for thermodynamics. Arch. Rational Mech. Anal. 54, 1–104.

Coleman, B.D., Owen, D.R., 1977. On thermodynamics of semi-systems with restrictions on the accessibility of states. Arch. Rational Mech. Anal. 66, 173–181.

Herrmann, H.J., Roux, S. (Eds.), 1990. Statistical Models of the Fracture of Disordered Media, North-Holland, Amsterdam.

Krajcinovic, D., 1996. Damage Mechanics. North-Holland, Amsterdam.

Krajcinovic, D., Mastilovic, S., Vujosevic, M., 1998. Brittle to quasi-brittle transition. Meccanica 33, 363–379.

Lemaitre, J., 1992. A Course on Damage Mechanics. Springer, Berlin.

Lubarda, V.A., Krajcinovic, D., 1995. Some fundamental issues in rate theory of damage elastoplasticity. Int. J. Plasticity 11, 763–797.

Mariano, P.M., 1997. Natural entropies in the state space of damaged bodies. Z. Angew. Math. Phys. ZAMP 48, 1002–1011.

Mariano, P.M., 1998. On the axioms of plasticity. Int. J. Solids Struct. 35, 1313–1324.

Mariano, P.M., 1999. Some remarks on the variational description of microcracked bodies. Int. J. Non-Linear Mech. 34, 633–642.

Mariano, P.M., Augusti, G., 1997. Some axioms and theorems in damage mechanics and fatigue of materials. Int. J. Solids Struct. 34, 3337–3350.

Noll, W., 1972. A new mathematical theory of simple materials. Arch. Rational Mech. Anal. 48, 1–50.

Owen, D.R., 1984. A frst course in mathematical foundations of thermodynamics. Springer, Berlin.

Šilhavý, M., 1989. Mass, internal energy and Cauchy's equations of motion in frame-indifferent thermodynamics. Arch. Rational Mech. Anal. 107, 1–22.

Šilhavý, M., 1997. The Mechanics and Thermodynamics of Continuous Bodies. Springer, Berlin.

Simo, J.C., Ju, J.W., 1987. Strain and stress-based continuum damage models. I. Formulation. Int. J. Solids Struct. 23, 821–840.